J. Appl. Maths Mechs Vol. 57, No. 1, pp. 87–93, 1993
Printed in Great Britain.

0021–8928/93 $24.00+.00

AN EFFECTIVE METHOD OF SOLVING CERTAIN TYPES OF INTEGRAL EQUATIONS IN PROBLEMS OF THE THEORY OF THE BENDING OF THIN PLATES†

V. B. ZELENTSOV

Rostov-on-Don

(*Received* 26 *November* 1991)

Integral equations (IE) to which one can reduce certain static and dynamic problems concerned with bending of thin plates with inclusions are considered. A method of solving these equations is proposed, which is based on a special approximation of the symbol of the kernel of the IE. The exact solution of the IE with an approximate kernel is constructed in the class of functions with integrable singularities at the ends of the integration interval.

OTHER methods of solving the integral equations (IE) of the mixed static problems of the theory of thin plate bending were considered in [1, 2] and elsewhere.

1. THE INTEGRAL EQUATION

We consider the following IE of the first kind, to which one can reduce some problems of the theory of the bending of the plates

$$\lambda^2 \int_{-1}^{1} \psi(\xi) k\left(\frac{\xi - x}{\lambda}\right) d\xi = 2\pi f(x), \quad |x| \leqslant 1 \tag{1.1}$$

The kernel of the IE is given by the formula

$$k(t) = \int_\Gamma K(u) e^{-iut} du, \; t = (\xi - x)/\lambda, \; \lambda = h/a \tag{1.2}$$

The function $K(u)$ in (1.2) is meromorphic, even and real-valued on the real axis, and has the following asymptotic properties

$$K(u) = u^{-3} + O(u^{-5}), u \to \infty; \quad K(u) = A + O(u^2), \; u \to 0 \tag{1.3}$$

In view of (1.3) the kernel is smooth and has the asymptotic forms

$$k(t) = \tfrac{1}{2} t^2 \ln|t| + O(t^2)$$

†*Prikl. Mat. Mekh.* Vol. 57, No. 1, pp. 77–82, 1993.

as $t \to 0$.

The integration contour Γ is chosen according to the limiting absorption principle [3, 4], since in dynamic problems some of the zeros and poles of $K(u)$ at a fixed frequency lie on the real axis. By choosing Γ one can construct the unique solution of a dynamic problem [4].

Since $K(u)$ is a meromorphic function in the complex plane $(u+\sigma+i\tau)$, it can be expressed in the form of an infinite product

$$K(u) = A \prod_{n=1}^{\infty} \frac{1+u^2\alpha_n^{-2}}{1+u^2\beta_n^{-2}}, \quad A = \lim_{u\to 0} K(u) \tag{1.4}$$

where $\pm i\alpha_n$ and $\pm i\beta_n$ are the zeros and poles of $K(u)$ in the complex plane.

2. APPROXIMATION OF THE SYMBOL OF THE KERNEL OF THE INTEGRAL EQUATION

For the solution of IE (1.1) the symbol of the kernel, i.e. the function $K(u)$, can be approximated by an expression of the special form

$$K(u) \approx \frac{\operatorname{th}(Au)}{u(u^2+p_0^2)} \prod_{n=1}^{N} \frac{u^2+q_n^2}{u^2+p_n^2} \tag{2.1}$$

which has the same asymptotic properties as $K(u)$ as $u \to 0$ and $u \to \infty$. The zeros q_n and poles p_n of (2.1), except for those lying on the real axis (on being computed with a given accuracy, they are substituted directly into (2.1)), can be chosen from the best approximation of $K(u)$ on the real axis and in a small neighbourhood of the real axis.

To solve Eq. (1.1), we make the substitution $u = \lambda u'$ and, after omitting the primes, obtain the equation

$$\int_{-1}^{1} \psi(\xi) k(\xi - x) d\xi = 2\pi f(x), \quad |x| \leqslant 1 \tag{2.2}$$

in which, taking the approximation into account, the kernel can be written in the form

$$k(t) = \int_{\Gamma} \frac{\operatorname{th}(A\lambda u)}{u(u^2+\gamma_0^2)} \prod_{n=1}^{N} \frac{u^2+\delta_n^2}{u^2+\gamma_n^2} e^{-iut} du, \quad \lambda\delta_n = q_n, \ \lambda\gamma_n = p_n \tag{2.3}$$

In what follows the notation

$$K(u) \approx \operatorname{th}(A\lambda u)/(u\xi(u^2)), \quad \xi(u^2) = L_2(u^2)/L_1(u^2) \tag{2.4}$$

will be used for convenience along with (2.3). Here $L_i(u^2)$ are polynomials of degree $2N$ and $2N+2$, respectively, and $\pm i\delta_n$ and $\pm i\gamma_n$ are the zeros of these polynomials, which coincide with the zeros and poles of the integrand in (2.3).

3. SOLUTION OF THE IE WITH KERNEL (2.3)

To solve IE (2.3) we write the formulae

$$\int_{-\infty}^{\infty} \psi(\beta) e^{-i\beta x} d\beta = \begin{cases} 2\pi\psi(x), & |x| \leqslant 1 \\ 0, & |x| > 1 \end{cases} \tag{3.1}$$

$$\psi(\beta) = \int_{-1}^{1} \psi(\xi) e^{i\beta\xi} d\xi$$

for the generalized Fourier integral transformation.

The solution of the original IE (2.2) with kernel (2.3) is sought for the right-hand side of the special form $f(x) = e^{-\epsilon x}$, for which it is required that $f(x)$ can be expanded into the Fourier series (generally speaking, ϵ is a complex number).

To begin with, we shall find [5] the solution of IE (2.2) for the special right-hand side $f_+(x) = \mathrm{ch}\epsilon x$. In this case, (2.2) takes the form

$$\int_{-1}^{1} \psi_+(\xi) k(\xi - x) d\xi = 2\pi \mathrm{ch}\epsilon x, \quad |x| \leqslant 1 \tag{3.2}$$

where $\psi_+(x)$ is the even part of $\psi(x)$.

The solutions of (3.2) are sought in the class of functions with integrable singularities at the end-points of the integration interval [1, 2]. Then it becomes necessary to regularize the integral in (3.2). It proves convenient to do this at a later stage of solving the equation. Equation (3.2) can now be represented as an ordinary differential equation (the notation (2.4) is used)

$$L_1(l)\Omega_+(x) = 2\pi L_2(l)\,\mathrm{ch}\epsilon x, \quad l = -d^2/dx^2, \quad |x| \leqslant 1 \tag{3.3}$$

$$\Omega_t(x) = \int_{-1}^{1} \psi_+(\xi) d\xi \int_{-\infty}^{\infty} \beta^{-1} \mathrm{th}(A\lambda\beta) \cos\beta(\xi - x) d\beta \tag{3.4}$$

where, in accordance with (2.4), $L_1(l)$ and $L_2(l)$ are differential operators of order $2N$ and $2N+2$ in x.

On inverting Eq. (3.3), we obtain the following new IE for determining $\psi_+(x)$

$$\Omega_+(x) = \chi(\epsilon, x), \; \chi(\epsilon, x) = 2\pi\xi(-\epsilon^2) + 2\pi \Sigma C_n \mathrm{ch}\delta_n x, \quad |x| \leqslant 1 \tag{3.5}$$

Here C_n is a system of constants, that remain unknown for the time being, and the summation is always from $n=1$ to $n=N$.

At this stage of solving the IE we regularize the outer integral in (3.5) using the class of functions

$$\psi_+(x) = \omega(x)(1 - x^2)^{-3/2}, \quad \omega(x) \in H_\gamma[-1,1], \quad \gamma < 1/2 \tag{3.6}$$

in which we seek the solution of the original IE [1, 2].

The feasibility of regularizing the integral in (3.5), rather than in (3.2), can be explained by the properties of the canonical regularization [6]. In view of the fact that $\psi_+(x)$ is an even function, the regularization in (3.5) can be carried out as follows: from $\psi_+(x)$ we subtract and then add to it the expression

$$\psi_+^0(x) = B_+(\mathrm{ch}\theta - \mathrm{ch}\theta x)^{-3/2}, \quad \theta = \pi/(A\lambda)$$

where B_+ is an unknown constant. Then we transform (3.5) to the form

$$\Omega_+^*(x) = 4\sqrt{2}\pi B_+ Q_{-1/2}^1(\mathrm{ch}\theta)/(\theta\,\mathrm{sh}\theta) + \chi(\epsilon, x), \quad |x| \leqslant 1 \tag{3.7}$$

where $\Omega_+{}^*(x)$ is an integral similar to (3.4) with $\psi_+(x)$ replaced by $\psi_+{}^* = \psi_+(\xi) + \psi_+^0(\xi)$.
It is seen that

$$\psi_+^*(x) = \omega(x)(1-x^2)^{-1/2},\quad \omega(x) \in H_\gamma[-1,1],\ \gamma > 0$$

i.e. the solution of IE (3.7) must belong to the class of functions with integrable singularities at the end-points of the integration interval.
Now, using the well-known results [5] for the Fourier transformation of the function $\psi_+(x)$

$$\Psi_+(\beta) = \int_{-1}^{1} \psi_+(\xi)\cos\beta\xi d\xi$$

we find that

$$\Psi_+(\beta) = 2\sqrt{2}\pi B_+\lambda^2 \frac{N(-1/2 + i\beta/\theta, -1/2)}{\theta \operatorname{sh}\theta\, Q_{-1/2}} + \Phi_+(\beta) \tag{3.8}$$

$$\Phi_+(\beta) = G_\epsilon P_{-1/2 + i\beta/\theta}(\operatorname{ch}\theta) - \pi^2\epsilon^2 \operatorname{sh}\theta H(-1/2 + i\beta/\theta, -1/2 + \epsilon/\theta) -$$
$$-\pi \operatorname{sh}\theta \Sigma C_n\delta_n^2 H(-1/2 + i\beta/\theta, -1/2 + \delta_n/\theta)$$
$$Q_{-1/2}G_\epsilon = \pi\xi(-\epsilon^2)\operatorname{sh}\theta H(-1/2 + \epsilon/\theta, -1/2) - \operatorname{sh}\theta \Sigma C_n N(-1/2 + \delta_n/\theta, -1/2)$$
$$H(u,v) = \theta^{-2}(P_u P_v^1 - P_v P_u^1)((u+1/2)^2 + (v+1/2)^2)^{-1}$$
$$N(u,v) = P_u Q_v^1 - Q_v P_u^1,\ P_v^\mu = P_v^\mu(\operatorname{ch}\theta),\ Q_v^\mu = Q_v^\mu(\operatorname{ch}\theta)$$

(P_v^μ and Q_v^μ are the associated spherical functions of the first and second kind).
The solution $\psi_+(x)$ is given by the formula

$$2\pi Q_{-1/2}\psi_+(x) = C_0\theta(\operatorname{sh}\theta Q_{-1/2} + r^2(1,x)Q_{-1/2}^1)/r^3(1,x) + 2\pi\varphi_+(x) \tag{3.9}$$
$$C_0 = 2\sqrt{2}\pi B_+/(\theta \operatorname{sh}\theta)$$
$$\varphi_+(x) = \xi(-\epsilon^2)F(-1/2, -1/2 + \epsilon/\theta, x) + \Sigma C_n F(-1/2, -1/2 + \delta_n/\theta, x)$$
$$F(u,v,x) = -\frac{\theta \operatorname{sh}\theta\, N(u,v)}{r(1,x)Q_u} - \theta^2(v+1/2)^2 \int_x^1 \frac{P_v(\operatorname{ch}\theta\tau)\operatorname{sh}\theta\tau}{r(\tau,x)} d\tau$$
$$r(u,v) = (2(\operatorname{ch}\theta u - \operatorname{ch}\theta v))^{1/2}$$

The constants $C_0, C_1, \ldots, C_n$ in (3.5) can be determined by substituting (3.8) into the dual IE, which is equivalent to (3.7). On taking the quadratures, the task of solving IE (3.2) can be reduced to the solution of the linear algebraic system

$$\sum_{k=0}^{N} x_k = f_m + \sum_{k=0}^{N} a_{mk}x_k,\quad m = 0, 1, \ldots, N \tag{3.10}$$
$$x_0 = -C_0 Q_{-1/2}^1,\ x_n = C_n P_{-1/2 + \delta_n/\theta} Q_{-1/2}^1.$$
$$a_{m0} = -Q_{-1/2}Q_{-1/2+\gamma_m/\theta}^1 / (Q_{-1/2}^1 Q_{-1/2+\gamma_m/\theta})$$
$$a_{mn} = Q_{-1/2}R(-1/2 + \delta_n/\theta, -1/2 + \gamma_m/\theta)/Q_{-1/2}^1$$
$$f_m = \pi\epsilon^2\xi(-\epsilon^2)E(-1/2 + \epsilon/\theta, -1/2 + \gamma_m/\theta, -1/2)/Q_{-1/2+\gamma_m/\theta}$$
$$R(u,v) = [(u+1/2)^2 P_u Q_v^1 - (v+1/2)^2 P_u^1 Q_v]/\Delta(u,v)$$
$$\Delta(u,v) = (u-v)(u+v-1)P_u Q_v$$
$$E(u,v,w) = Q_w T(u,v) - Q_v T(u,w),\quad T(u,v) = \theta^2 N(u,v)[(u+1/2)^2 - (v+1/2)^2]^{-1}$$

As in the even case considered above, to construct a solution of IE (1.1) with an odd right-hand side of the special form $f_-(x) = \text{sh}\epsilon x$, one can change from (1.1) to an equation of the type (3.2) by making the same substitution, which in the case in hand leads to the equation

$$\int_{-1}^{1} \psi_-(\xi) k(\xi - x) d\xi = 2\pi \text{sh}\epsilon x, \quad |x| \leqslant 1 \tag{3.11}$$

where $k(t)$ is given by (2.3) and where $\psi_-(x)$ is the odd part of $\psi(x)$.

The solution $\psi_-(x)$ of (3.11) is sought in the class (3.6) of functions with non-integrable singularities at the end-points of the integration interval. To this end one can use essentially the same scheme for solving (3.11) as in the even case considered above. As a result, (3.11) can be reduced to solving an equation that differs from (3.5) by the fact that ψ_t is replaced by ψ_-, $\text{ch}\epsilon x$ is replaced by $\text{sh}\epsilon x$ and $\text{ch}\delta_n x$ by $\text{sh}\delta_n x$. The constants C_n will then be replaced by new constants D_n. Furthermore, the integral must also be regularized by subtracting from $\psi_-(x)$ and then adding to it the expression $B_-\text{sh}\theta x(\text{ch}\theta - \text{ch}\theta x)^{-3/2}$, in which B_- is an unknown constant.

The solution $\psi_-(x)$ has the form

$$\psi_-(x) = -D_0\theta^2 \text{sh}\theta x / [8\pi r^3(1, x)] + \varphi_-(x) \tag{3.12}$$

$$\varphi_-(x) = \zeta(-\epsilon^2)\text{sh}\theta x [S(-½ + \epsilon/\theta, x) - \Sigma D_n S(-½ + \delta_n/\theta, x)]$$

$$S(u, x) = \theta(u + ½)\left[\frac{P_u}{r(1,x)} - \theta \int_x^1 \frac{P_u(\text{ch}\theta\tau)}{r(\tau, x)} d\tau\right]$$

Hence we have obtained the solutions of the original IE for the special even and odd right-hand sides (3.9) and (3.12). The solution of the IE with right-hand side $f(x) = e^{-\epsilon x}$ can be constructed from the formula $\psi(x) = \psi_t(x) - \psi_-(x)$. In doing so one must not forget the connection between p_n, q_n and δ_n, γ_n in (2.3) in the final construction of the solution.

In what follows, when considering specific examples of problems which can be reduced to IE (1.1), we shall need formulae for the solution of this equation in the special case when the right-hand side is equal to one. In this case, to obtain the solution it suffices to pass to the limit as $\epsilon \to 0$ in formulae (3.9) and (3.10) for the even case. As a result, we shall obtain a simpler solution for $f(x) = 1$, in which the first term in the expression for $\varphi_+(x)$ is equal to zero, and $f_m = 1$.

4. PROBLEMS REDUCIBLE TO IE (1.1) AND SOME SUPPLEMENTARY FORMULAE

As a first example we consider the problem of oscillations of a thin rectilinear rigid inclusion of length $2a$ in a pre-stressed Kirchhoff-Love plate [7]. The inclusion oscillates due to a force that varies according to the harmonic law $Pe^{-i\omega t}$. The plate can be represented as a strip of width $2h$. The inclusion is placed in the middle of the plate parallel to its side edges, which are rigidly fixed.

With the aid of the Fourier integral transformation this problem can be reduced to the solution of IE (1.1) with kernel $k(t)$ whose symbol has the form

$$K(u) = 2\frac{\sigma_1\sigma_2(1 - \text{ch}\sigma_1\text{ch}\sigma_2) + ½(\sigma_1^2 + \sigma_2^2)\text{sh}\sigma_1\text{sh}\sigma_2}{\tau\sigma_1\sigma_2(\sigma_2\text{ch}\sigma_1\text{sh}\sigma_2 - \sigma_1\text{sh}\sigma_1\text{ch}\sigma_2} \tag{4.1}$$

$$\sigma_{1,2} = (u^2 + q_{1,2}^2 \mp \tau)^{½}, \quad \tau = [2(q_2 - q_1)u^2 + q_2^2 + \kappa^4]^{½}$$

$$\kappa^4 = \rho\omega^2 h^2 D^{-1}, \quad q_i = ½N_{ii}h^2 D^{-1}, \quad i = 1, 2$$

Here q_i are the magnitudes of the generalized prestresses at the middle surface of the plate, k is the generalized frequency of oscillations, and N_{ii} are the forces on the middle surface of the plate. Since the inclusion is rectilinear, $f(x)=1$ in IE (1.1).

As another example, we consider a problem whose formulation differs from that of the former one only by the fact that side edges of the plate are supported by hinges. In the same way as above, the problem can be reduced to the solution of IE (1.1) with the symbol of the kernel

$$K(u) = \tau^{-1} (\sigma_2^{-1} \operatorname{th}\sigma_2 - \sigma_1^{-1} \operatorname{th}\sigma_1) \tag{4.2}$$

The simplified formulae mentioned at the end of Sec. 3, which determine the reactive shear force arising on the rigid inclusion in the plate, have been used to solve the resulting equations.

The effect of the prestress q_2 ($q_1=0$), the oscillation frequency κ, and the conditions for fixing the side edges of the plate on the amplitude of the force P acting on the inclusion (corresponding to the unit displacement of the rigid inclusion) has been studied with the aid of the above-mentioned formulae.

The functions $K(u)$ ((5.1), (5.2)) for the problems considered have all the required properties (Secs 1 and 2), except for being meromorphic, since they have branching points on the imaginary axis in the complex plane $u=\sigma+i\tau$. This is not an obstacle as far as the method of solving the IE proposed in this paper is concerned, because the functions $K(u)$ can be approximated by expressions of the form (2.1) on the real axis.

In the case under consideration, the amplitude of the force acting on the inclusion per unit length of the inclusion has the form

$$P = \int_{-1}^{1} \psi_+(\xi)\,d\xi = \tfrac{1}{2} C_0 N(-\tfrac{1}{2}, -\tfrac{1}{2})/Q_{-\frac{1}{2}} + \xi(-\epsilon^2)\,\Upsilon(-\tfrac{1}{2}+\epsilon/\theta, -\tfrac{1}{2}) + \Sigma C_n \Upsilon(-\tfrac{1}{2} + \tag{4.3}$$

$$+ \delta_n/\theta\ -\tfrac{1}{2})$$

$$\Upsilon(u, v) = \tfrac{1}{2}\pi[\theta P_u(u+\tfrac{1}{2}) + \operatorname{sh}\theta\, N(u, v)\,(u-1)^{-1}]/Q_{-\frac{1}{2}}$$

In the case of a rectilinear inclusion $f(x)=1$ under consideration, the formula can be simplified, since one must set $\epsilon=0$ in (4.3).

The dependence of the reduced force P, that acts on the inclusion and pushes it down to unit depth, on the generalized frequency κ and the generalized preliminary compression–extension q_2 is presented in Fig. 1 by solid lines for the first problem, and by dashed lines for the second problem. The geometric

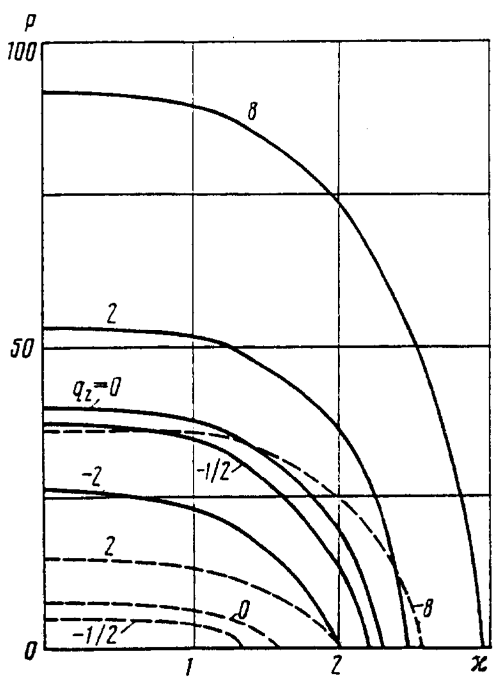

FIG. 1.

parameter is $\lambda = h/a = 1$. It clearly follows from the graphs that the force needed to push the inclusion down to unit depth must be greater for the first problem than for the second. Near the first resonance frequency the plate becomes more susceptible to bending and the force decreases (tends to zero) in both problems. The graphs also indicate that in the case of the first problem the system is more rigid and has higher characteristic frequencies than in the case of the second problem. It can also be seen that by changing the preliminary compression–extension, one can change the characteristic oscillation frequencies of the plate. The prestresses have a significant effect on the magnitude of the force, which decreases as the preliminary compression increases, and increases as the preliminary extension increases.

As has been mentioned above, it is important to study the phase velocities of the displacements of the plate both for practical reasons and in order to construct the correct solution of IE [4]. Using a simple example of the second problem in which the function $K(u)$ is defined by (4.2), we will demonstrate the dependence of the spectrum of poles on the parameters. In the plane $u^2 = y$, $\kappa^2 = x$ the poles of $K(u)$ from (4.2) are given by the system of hyperbolae

$$
\begin{gathered}
x^2/a_k^2 - (y + c_k)^2/a_k^2 = 1 \\
c_k = q_2^2 - (q_2 - q_1) + \pi^2 (1/2 + k)^2, \quad a_k^2 = (q_2 - q_1)(q_2^2 + c_k^2) - q_2, \quad k = 0, 1, 2, \ldots
\end{gathered}
\tag{4.4}
$$

In other, more complicated cases, they have been obtained numerically.

The numerical realization of the method proposed for solving integral equations of the type (1.1) is highly efficient over practically the entire range of variation of $\lambda \in (0, \infty)$. An important advantage of the method is the absence of singular quadratures in the resulting solution, that often arise when solving the problems from the class under consideration. The error of the resulting solutions does not exceed the approximation error for the symbol of the kernel of the IE.

The author wishes to express his gratitude to V. M. Aleksandrov for his interest in this work.

REFERENCES

1. ONISHCHUK O. V. and POPOV G. Ya., Some problems concerned with the bending of plates with cracks and thin inclusions. *Izv. Akad. Nauk SSSR, MTT* **4**, 141–150, 1980.
2. ZELENTSOV V. B., Solution of certain integral equations in mixed problems of the theory of plate bending. *Prikl. Mat. Mekh.* **48**, 983–991, 1984.
3. BABESHKO V. A., Radiation conditions for an elastic layer. *Dokl. Akad. Nauk SSSR* **213**, 547–549, 1973.
4. VOROVICH I. I. and BABESHKO V. A., *Mixed Dynamic Problems of the Theory of Elasticity for Non-classical Domains.* Nauka, Moscow, 1979.
5. ZELENTSOV V. B., Solution of a class of integral equations. *Prikl. Mat. Mekh.* **46**, 815–820, 1982.
6. VILENKIN N. A., GORIN E. A., KOSTYUCHENKO A. G. *et al.*, *Functional Analysis.* Nauka, Moscow, 1954.
7. DONNELL L. G., *Beams, Plates and Shells.* Nauka, Moscow, 1982.

Translated by T.J.Z.